

Covariant Calculus for Effective String Theories

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A covariant calculus for the construction of effective string theories is developed. Effective string theory, describing quantum string-like excitations in arbitrary dimension, has in the past been constructed using the principles of conformal field theory, but not in a systematic way. Using the freedom of choice of field definition, a particular field definition is made in a systematic way to allow an explicit construction of effective string theories with manifest exact conformal symmetry. The impossibility of a manifestly invariant description of the Polchinski-Strominger Lagrangian is demonstrated and its meaning is explained.

I. INTRODUCTION

Although fundamental string theory is of course confined to certain critical dimensions, string-like phenomena do indeed appear as defects, solitons or effective descriptions in a variety of physical situations. Since these situations generically are of non-critical dimension, an effective theory of strings must exist in order to describe them.

Polchinski and Strominger (PS) proposed the construction of such a theory in [1]. As in other constructions of effective theories, the formulation is required to exhibit the correct symmetries, while dropping such requirements as renormalisability and polynomial lagrangian, which are usually taken as minimal for a ‘fundamental’ theory expected to be valid at all energy scales. In particular, PS treated an expansion around a long-string vacuum, where the characteristic string length R is taken as a large parameter. The effective action is thus expanded in inverse powers of R . The notable difference with fundamental string theory is that the effective PS theory contains a variable central charge, which can be adjusted for consistency in any dimension. Although Polchinski and Strominger showed that the price one has to pay for this quantum consistency in any dimension is the allowance of nonpolynomial terms in the action, in such a perturbative expansion around the long-string vacuum, such terms are perfectly acceptable.

PS were able to calculate the excitation spectrum including in the effective action the first correction after the leading Polyakov-type (equivalently Nambu-Goto) term. Surprisingly, the spectrum does not deviate from that of Nambu-Goto theory at this order. It has been shown in [2] and [3] using an action valid to order R^{-3} , that at even the next relevant order after this, the spectrum does not differ from that of Nambu-Goto theory.

In the original formulation [1] of PS, the choice was made to omit terms in the effective action proportional to the leading-order equations of motion (EOM), which may be removed to appropriate order by a field redefinition. In fact, as we have shown in [3], dropping or including a particular set of such ‘irrelevant terms’¹ amounts to a particular choice of field definition; the PS field definition corresponds to the omission of all EOM terms. It was pointed out in [3] that different such choices of field definition will correspond to actions invariant under different transformation laws; different field definitions are related by some field redefinition transformation, and this of course relates potentially different transformation laws, each representing the conformal symmetry of the theory.

The effective action proposed by Polchinski and Strominger is

$$S_{PS} = \frac{1}{4\pi} \int d\tau^+ d\tau^- \left\{ \frac{1}{a^2} \partial_+ X^\mu \partial_- X_\mu + \beta \frac{\partial_+^2 X \cdot \partial_- X \partial_+ X \cdot \partial_-^2 X}{(\partial_+ X \cdot \partial_- X)^2} + \mathcal{O}(R^{-3}) \right\}. \quad (1)$$

The quantity R signifies the length of the string and in what follows, is taken to be large. Consideration is restricted to fluctuations around the classical background. The leading-order equation of motion $\partial_+ \partial_- X^\mu = 0$ has the solution $X_{cl}^\mu = e_+^\mu R\tau^+ + e_-^\mu R\tau^-$, where $e_-^2 = e_+^2 = 0$ and $e_+ \cdot e_- = -1/2$.

The action of eqn.(1) is invariant, to the appropriate order, under the transformation

$$\delta_-^{\text{PS}} X^\mu = \epsilon^-(\tau^-) \partial_- X^\mu - \frac{\beta a^2}{2} \partial_-^2 \epsilon^-(\tau^-) \frac{\partial_+ X^\mu}{\partial_+ X \cdot \partial_- X} \quad (2)$$

(and another: $\delta_+ X$ with $+$ and $-$ interchanged).

PS proposed an algorithm for extending their analysis to higher orders which can be stated as follows. Firstly,

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¹ We retain the terminology used in [1, 3], whereby terms proportional to the leading-order EOM are called *irrelevant*, and terms irreducible to this form are deemed *relevant*.

write down all possible (1,1) terms which according to PS simply amounts to keeping terms whose net number of \pm -derivatives (terms in the denominator count negatively) is (1,1). Secondly, discard all terms proportional to the leading-order constraints $\partial_{\pm}X \cdot \partial_{\pm}X$ and their derivatives. Finally, use integration by parts to relate equivalent terms.

At this point one will have terms with and without ‘mixed derivatives’, terms sporting mixed derivatives being what we have called *irrelevant* in [3]. The PS prescription then is to discard all irrelevant terms and *find transformation laws* that leave the relevant terms in the action invariant.

Clearly, generalisation of the PS formalism requires not only finding the right action to the desired order, but also determining the appropriate transformation laws. This is reminiscent of the early days of supergravity theories, and this procedure becomes tedious and unwieldy with increasing order. Not only does the procedure become tedious, more importantly it does not lend itself to a systematic method of construction and analysis at higher orders. It is the purpose of the present paper to propose a simplified formalism, in both a technical and conceptual sense. We propose to achieve this through a formulation wherein the transformation laws are independent of the particular action chosen. We start by demonstrating how this can be done for the PS action itself.

Recall that the PS proposal for the leading correction was based on a comparison with the Liouville action for subcritical strings

$$S_{Liou} = \frac{26-D}{48\pi} \int d\tau^+ d\tau^- \partial_+ \phi \partial_- \phi \quad (3)$$

They argued that in effective string theories the conformal factor e^{ϕ} should be replaced by the component $\partial_+X \cdot \partial_-X$ (in the conformal gauge) of the induced metric on the worldsheet. They had also proposed replacing $(26-D)/12$ by a parameter β which was to be determined by requiring the vanishing of the total central charge in all dimensions, though they eventually found β to be just the same as in the Liouville theory². A direct application of this idea would have suggested the total action

$$S_2 = \frac{1}{4\pi} \int d\tau^+ d\tau^- \left\{ \frac{1}{a^2} \partial_+ X^\mu \partial_- X_\mu + \beta \frac{\partial_+(\partial_+X \cdot \partial_-X) \partial_-(\partial_+X \cdot \partial_-X)}{(\partial_+X \cdot \partial_-X)^2} \right\}. \quad (4)$$

It is easily shown that $S_{(2)}$ is invariant under the transformations

$$\delta_{\pm}^0 X^\mu = \epsilon^{\pm} (\tau^{\pm}) \partial_{\pm} X^\mu. \quad (5)$$

For the purposes of the present discussion, we consider the ‘-’ alternative, without loss of generality. More explicitly, if we write the second part of S_2 , $S_2^{(2)}$, as

$$S_2^{(2)} = \int d\tau^+ d\tau^- L_2 \quad (6)$$

it is easy to show that

$$\delta_-^0 L_2 = \partial_- (\epsilon^- L_2) + \partial_-^2 \epsilon^- \partial_+ L \quad (7)$$

The first term is what one would have expected if L_2 had transformed as a scalar density, and the second term is a departure from this. We shall explain this important point later; for the moment it suffices to note that the additional term can be rewritten as

$$\partial_-^2 \epsilon^- \partial_+ L = \partial_+ (\partial_-^2 \epsilon^- L) \quad (8)$$

ensuring the invariance of S_2 if we neglect integrals of total derivative terms.

Polchinski and Strominger [1] build their effective action while discarding all total derivatives. This has generally been done in the literature; treatment of total derivative terms in the action is a subtle and important issue that in principle needs careful scrutiny. In this paper, we shall nevertheless proceed with the premise that such total derivative terms can be ignored.

The algebra of the PS transformations of eqn.(2) is

$$[\delta_-^{PS}(\epsilon_1^-), \delta_-^{PS}(\epsilon_2^-)] = \delta_-^{PS}(\epsilon_{12}^-) + \mathcal{O}(R^{-4}), \quad (9)$$

where $\epsilon_{12}^- = \epsilon_1^- \partial_- \epsilon_2^- - \epsilon_2^- \partial_- \epsilon_1^-$. On the other hand, the algebra of the transformations of eqn.(5) is

$$[\delta_-^0(\epsilon_1^-), \delta_-^0(\epsilon_2^-)] = \delta_-^0(\epsilon_{12}^-). \quad (10)$$

Thus both generate the same group of symmetry transformations, namely the conformal group. While the PS transformations realise this only approximately, to $\mathcal{O}(R^{-4})$ which however is sufficient in context as the PS action is defined to $\mathcal{O}(R^{-3})$, the transformations (5) leaving S_2 invariant do so *exactly*.

It should be noted that field redefinitions do not change the algebra of transformations, though the transformation laws are themselves changed. This is indeed what is happening here and to understand this note

$$S_{PS} - S_2 = \frac{\beta}{4\pi} \int L^{-2} \partial_+ X \cdot \partial_- X \partial_+ L \quad (11)$$

where $L = \partial_+X \cdot \partial_-X$. Thus the additional terms are proportional to the EOM of the leading part of the action, and can be removed through field redefinitions to appropriate order. A detailed discussion of how the field redefinition corresponding to eqn.(11) indeed connects eqn.(2) and eqn.(5), as well as an alternate description of effective string theories based on the action S_2 can be found in [3].

² NDH thanks Hikaru Kawai regarding why this has to be so.

II. SIMPLIFIED FORMALISM

The above discussion points to a much simpler formulation of effective string theories whereby the transformation law is always of the form (5). Furthermore, S_2 provides an example of an effective string theory which is in principle valid to all orders in $1/R$, and would provide an important test case for understanding higher-order corrections to the spectrum of Nambu-Goto theory. Before beginning construction, we briefly discuss here the merits of such a formulation.

A. Covariant Formalism

Since the transformation is fixed and does not have to be fine-tuned to a given action we now have the possibility of a systematic covariant calculus for the construction of invariant actions. One is always assured of obtaining invariant actions this way whereas a generalisation of the PS algorithm based on the naïve (1,1) counting there is no guarantee that to a given action one can always find a suitable transformation law. In a given case one has either to use trial and error or to identify a field redefinition, and even then the results are valid only up to some prescribed order. On the other hand each construction based on a covariant calculus will yield actions valid to all orders in $1/R$. In the present paper, we approach the construction of a covariant formulation in two ways, explicitly given in sections [IV] and [V].

B. Measures and Quantum Equivalences

An important issue tied up with field redefinitions in Quantum Field Theory is that of the quantum equivalence of theories related by them. This has been addressed to some extent in [3]. In the path integral formulation, to which the canonical formulation should be equivalent after all care has been exercised, this concerns the invariant (under the symmetry transformations) measure to be adopted as well as its transformation under field redefinitions. Both these issues are naturally taken care of in a covariant formulation based on eqn.(5). As the ‘naïve’ measure $\prod_{\sigma} dX(\sigma)$ is invariant modulo irrelevant regularisation-dependent factors, specifying the action specifies everything. This is a great simplification. Covariance fixes the irrelevant terms also thereby fixing a field definition also.

The last point also means that in the covariant formulation one cannot simply drop the irrelevant terms as that would amount to changing the field definition which generically would result in a change of measure, as well as the transformation law which would necessitate changing the covariant calculus itself. If, however, it can be shown that the resulting field redefinition to a certain order does not spoil quantum equivalence (i.e. the measure is left unchanged to the relevant order), irrelevant terms can

indeed be dropped. However, the resulting changes in the transformation law have to be taken into account.

III. TWO PATHS TO COVARIANCE

A. Symmetry Content

It is the symmetry content of the theory, more precisely the symmetry variations (transformation laws) that determine the covariant calculus. Clearly, this is dictated by the physics of the system and is not a matter of formalism. We are seeking a covariant formalism for the symmetry variations of eqn.(5). Before doing so it is worthwhile understanding why these should embody the symmetry content of effective string theories. Justifiably one could have taken the view that this depends on the details of systems with effective string behaviour. For example, it may have been so that only the ‘global’ version of eqn.(2) as against the more restrictive ‘local’ version correctly captures the relevant symmetry. It just so happens that for both the leading order effective action as well as for the PS terms, the global invariance also implies the local invariance. Clearly at high orders this will no longer be true. Then it will become a matter of ‘phenomenology’ to find out which will be a better description. Nevertheless, we shall develop a covariant formulation for the local invariance. Should phenomenology prefer the global invariance as the true symmetry the rationale for such a covariant formulation would be considerably weakened. It should be pointed out that even then such a covariant formulation will be useful as a framework for any systematic phenomenological analysis.

In what follows we shall actually seek something more general. We shall seek the most general coordinate invariant version of the transformation laws of eqn.(5) and develop the corresponding covariant calculus.

B. Two paths

We have mentioned that we will construct our covariant formalism in two alternate ways. The two distinct approaches are similar to what has been followed in the case of fundamental string theories.

The first, the Nambu-Goto method, is to start with the action

$$S_{\text{NG}} = \int \sqrt{\det(\partial_{\alpha} X \cdot \partial_{\beta} X)}, \quad (12)$$

invariant under eqn.(35). This approach is characterised by the absence of an intrinsic metric on the worldsheet. The composite operator, $\partial_{\alpha} X \cdot \partial_{\beta} X$, also the induced metric on the worldsheet due to the flat geometry of the target space, transforming exactly as the metric, acts as a substitute metric in realising general coordinate invariance.

The second, the Polyakov method, introduces an auxiliary metric field $h_{\alpha\beta}$. The action equivalent to eqn.(12) is the Polyakov action

$$S = \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X, \quad (13)$$

invariant under eqn.(35) and eqn.(36). The metric field $h_{\alpha\beta}$ is independent. The Polyakov action is also general-coordinate invariant, although the real symmetry content is reflected in the invariance of the action under the Weyl transformations, $h_{\alpha\beta} \rightarrow \lambda(\sigma) h_{\alpha\beta}$.

In effective string theories one will necessarily have to consider higher derivative terms in the action and these may not in general be Weyl invariant. This will require some additional technical structures which are developed in [V]. In fundamental string theories one did not need these.

Although conceptually distinct, both these approaches lead to identical content for the final effective string theory they are designed to produce. This will be shown in detail in section [VI].

C. Reparametrisation Invariance vs. Symmetry

It can be seen from the above that reparametrisation invariance plays radically different rôles in the two approaches. It is worthwhile understanding this important difference. Generically reparametrisation invariance is considered in situations with an intrinsic metric for the space-time manifold. In such situations, any action can be made reparametrisation invariant, and consequently the latter is devoid of physical content. It is only the statement that specific choice of coordinates is immaterial, and that it is desirable to write the theory in a *form* which reflects this. It is not a symmetry of the physical system.³ This is best illustrated by the following elementary example. Consider a theory with a scalar field ϕ on a flat background. The action could look something like

$$S = \int d^d x (\partial_\mu \phi \partial_\mu \phi - m^2 \phi^2 + \dots + \phi^4 + \dots). \quad (14)$$

Since it doesn't matter what coordinates we choose, it is desirable to represent the theory in a way that under general coordinate transformations, the action is invariant, in the sense that it *has the same form in any coordinate system*:

$$S = \int d^d x \sqrt{g} (g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - m^2 \phi^2 + \dots + \phi^4 + \dots). \quad (15)$$

It should be noted that this does not *change* the theory at all. The crucial point is that this applies to *any* theory.

³ Almost immediately after Einstein had formulated his General Theory of Relativity, Kretschmann [4] had pointed this out but it is not widely appreciated even at present.

Any theory can thus be written in such diffeomorphism-invariant form, so that a particular choice of coordinates can be postponed or avoided.

Turning our attention to symmetries, the situation is conceptually different. Symmetries, unlike diffeomorphisms in the above context, restrict the physical content of the theory and not any theory can be made invariant under the symmetry transformations. A trivial example in the context of the above mentioned scalar field theory is the symmetry under $\phi \rightarrow -\phi$. This restricts the form of the action to have only even powers of ϕ and not all actions possess this feature irrespective of the choice of coordinates.

While any *theory* must be diffeomorphism invariant, and therefore can be written down in a covariant way which reflects this, only certain theories have a particular symmetry. There is no way to take an arbitrary theory and somehow *make* it symmetric.

D. Conformal Symmetry from Reparametrisation Invariance

As will be seen later, in the first approach conformal symmetry emerges as residual invariance of the conformal gauge choice eqn.(27). Since in this approach this symmetry arises from the underlying reparametrisation invariance, which has been argued above to be generically void of physical content, it is important to understand the precise connection between this emergent conformal symmetry and reparametrisation invariance.

Does the group of reparametrisations contain the group of conformal transformations? Strictly, it does indeed; A mapping (assumed invertible, differentiable, etc..) from x coordinates to x' is a general coordinate transformation, and of course the mappings which correspond to conformal transformations are of the same kind. It is crucial to realise that reparametrisation invariance does not always result in conformal symmetry upon choosing the conformal gauge.

This is best exemplified again by the scalar field example of eqn.(15) in two dimensions where coordinates can generically (at least locally) be chosen so that the intrinsic metric is of the form

$$g_{\alpha\beta} = \begin{bmatrix} 0 & \varphi \\ \varphi & 0 \end{bmatrix} \quad (16)$$

in coordinates σ^\pm . This does not use up all available freedom, and residual coordinate transformations which preserve this form are easily seen to comprise the conformal group. The action of eqn.(15) is indeed invariant under the action of these transformations; yet the physical content of the theory is exactly that of eqn.(14). What is more, *any* scalar field theory can be made to have this invariance, and it therefore does not represent any physical symmetry.

In the second approach what does represent a symmetry is the invariance under Weyl-scaling. In the scalar

field example also one sees that not all actions possess this invariance in keeping with what a symmetry is.

In our first approach, in which we do not treat the metric as an independent field, we do not make any assumption of Weyl symmetry. Nevertheless there is a symmetry in this case and that is traceable entirely to reparametrisation invariance. The rôle of an intrinsic metric is instead played by suitable composite fields constructed out of the physical fields. The only degrees of freedom in the theory are taken to be the X scalar fields. Since now not every action can be reparametrisation invariant, reparametrisation invariance in this case becomes a physical symmetry.

Going to the equivalent of (16) by a coordinate choice, where $g_{\alpha\beta}$ is now a composite field transforming like a metric, and choosing

$$\varphi \equiv \partial_+ X \partial_- X$$

one realises the conformal gauge of the first formalism with conformal invariance as the residual symmetry.

Thus in both approaches conformal invariance emerges as the residual invariance of the conformal gauge; but in the first case it emerges as a true physical symmetry, while in the second approach it is like a generic reparametrisation invariance but not a symmetry. It is the underlying Weyl-scaling invariance that finally results in the conformal invariance being elevated to a symmetry in the precise sense that not all actions are invariant. It should also be emphasised that in other gauges, like for example the transverse gauge $X^0 = \tau$, $X^1 = \sigma$ there will be nothing like conformal symmetry in either of the two approaches. In that sense, this is true of fundamental string theory also, there is nothing intrinsic to conformal symmetry per se; what is important is the symmetry content of the gauge-unfixed theory.

E. The Denominator Principle

What is being developed in this work is for effective string theories as opposed to fundamental string theories. The allowed actions for effective string theories can sometimes become singular for certain string configurations but for long strings fluctuating about a classical background such action terms should be sensible. However even this requirement should preclude terms in effective string actions whose denominators can become singular for some fluctuation of the effective string. This becomes an important guiding principle for effective string theories. In particular, it needs to be evoked while restricting substitute metrics in section [IV], the Weyl connection in section [V] as well as restricting the Weyl-weight compensators in section [VD].

IV. COVARIANT CALCULUS I: NON-INTRINSIC METRIC

In this section we make one of our proposals for a covariant formulation. To attain final covariance under conformal transformations, we shall use initially covariance under worldsheet general coordinate transformations only. *A priori*, a metric field is needed for any covariant formulation. In the spirit of PS we shall not introduce any intrinsic metric on the worldsheet. It suffices to have an object that transforms the same way as a metric under general coordinate transformations. One natural choice for such a *metric substitute* is the *induced metric* on the worldsheet

$$g_{\alpha\beta} = \partial_\alpha X \cdot \partial_\beta X. \quad (17)$$

Strictly speaking, any quantity built out of the basic variables X^μ with the correct $2-d$ tensor structure is also a *bona fide* candidate. In fact, any such object would lead to a formulation in which covariance is manifest, and effective actions could be constructed. The choice of eqn.(17) is in a sense the *simplest* one can make and it is also the choice that PS made.

Finally, the quantity we choose here will later appear, in gauge-fixed form, in various denominators. As we require the effective theory to be valid on any fluctuation, eqn.(17) is the simplest choice, just as L was for denominators in the initial PS formulation and subsequent treatments [2, 3]. These choices are also consistent with the Denominator Principle enunciated above. All other choices, upon resorting to perturbation in R^{-1} , are essentially equivalent to this.

Once the metric substitute is chosen, the rest of the construction is along standard lines of Riemannian Geometry. Various covariant derivatives $D_{\alpha\beta\gamma\dots}X$ can be written, and invariants made out of the g and these objects. In addition, tensors containing only the derivatives of g can only enter through the Riemann curvature tensor $R_{\alpha\beta\gamma\delta}$ and its covariant derivatives. Since in two dimensions

$$R_{\alpha\beta\gamma\delta} = (g_{\alpha\gamma}g_{\beta\delta} - g_{\beta\gamma}g_{\alpha\delta})\frac{R}{2}, \quad (18)$$

where R is the Ricci scalar, one need consider R and its covariant derivatives only. This vastly simplifies the construction of actions.

A. Some Manifestly Covariant Actions - I

In this section we provide a few examples of manifestly covariant action terms, more specifically, terms that transform as scalar densities. A systematic procedure for construction of such terms to any desired order in $1/R$ will be given later, in section [VII]. One could

begin with

$$I_{\text{cov}} = \sqrt{g} D_{\alpha_1 \beta_1 \dots} X^{\mu_1} D_{\alpha_2 \beta_2 \dots} X^{\mu_2} \cdot A^{\alpha_1 \beta_1 \dots \alpha_2 \beta_2 \dots} B_{\mu_1 \mu_2 \dots} \quad (19)$$

where $A^{\alpha_1 \beta_1 \dots \alpha_2 \beta_2 \dots}$ is composed of suitable factors of Levi-Civita and metric tensors on the two-dimensional worldsheet and $B_{\mu_1 \mu_2 \dots}$ made up of $\eta_{\mu\nu}$ and Levi-Civita tensors in target space. In the conformal gauge this construction can be done even more simply by stringing together a number of covariant derivatives so that there are equal net numbers of $(+, -)$ indices, and finally use sufficient inverse powers of L to make the expression $(1, 1)$.

Now we illustrate these methods by covariantising some terms proposed by Drummond. The PS term itself is at leading order R^{-2} , and Drummond [2] found four possibilities for the next relevant order- R^{-6} part of the action. These are

$$M_1 = \frac{1}{L^3} \partial_+^2 X \cdot \partial_+^2 X \partial_-^2 X \cdot \partial_-^2 X, \quad (20)$$

$$M_2 = \frac{1}{L^3} \partial_+^2 X \cdot \partial_-^2 X \partial_+^2 X \cdot \partial_-^2 X, \quad (21)$$

$$M_3 = \frac{1}{L^4} \partial_-^2 X \cdot \partial_+^2 X \partial_- X \cdot \partial_+^2 X \partial_-^2 X \cdot \partial_+ X, \quad (22)$$

$$M_4 = \frac{1}{L^5} (\partial_- X \cdot \partial_+^2 X)^2 (\partial_-^2 X \cdot \partial_+ X)^2. \quad (23)$$

Considering the first two terms, we can expect these to be contained in the covariant forms

$$\mathcal{M}_1 = \sqrt{g} D_{\alpha_1 \beta_1} X \cdot D_{\alpha_2 \beta_2} X D^{\alpha_1 \beta_1} X \cdot D^{\alpha_2 \beta_2} X \quad (24)$$

$$\mathcal{M}_2 = \sqrt{g} D_{\alpha_1 \beta_1} X \cdot D^{\alpha_1 \beta_1} X D_{\alpha_2 \beta_2} X \cdot D^{\alpha_2 \beta_2} X \quad (25)$$

B. Conformal Gauge and Conformal Transformations in Calculus-I

The PS formulation specifically hinged on the use of the conformal transformations

$$\tau^\pm \rightarrow \tau^\pm + \epsilon^\pm; \quad \partial_\pm \epsilon^\mp = 0 \quad (26)$$

In the context of general coordinate invariance, these transformations arise as the residual transformations maintaining the conformal gauge

$$g_{++} = g_{--} = 0 \quad (27)$$

In this gauge $g_{+-} = g_{-+} = L$ transforms as a true $(1, 1)$ -tensor under the conformal transformations. Importantly, $g^{+-} = g^{-+} = L^{-1}$ transforms as a $(-1, -1)$ tensor. It is straightforward to work out the non-vanishing components of the Christoffel connection as well as the Riemann curvature tensor:

$$\Gamma_{++}^{(1)+} = \partial_+ \ln L; \quad \Gamma_{--}^{(1)-} = \partial_- \ln L \quad (28)$$

$$\begin{aligned} R_{+-+}^+ &= -R_{++-}^+ = \partial_+ \partial_- \ln L \\ R_{-+-}^- &= -R_{--+}^- = \partial_+ \partial_- \ln L \end{aligned} \quad (29)$$

All the remaining components are zero. The resulting scalar curvature is

$$R = -2 \frac{\partial_+ \partial_- \ln L}{L}; \quad \sqrt{g} R = -2 \partial_+ \partial_- \ln L \quad (30)$$

We next give explicit expressions for some covariant derivatives:

$$\begin{aligned} D_\pm X^\mu &= \partial_\pm X^\mu \\ D_{++} X^\mu &= \partial_{++} X^\mu - \partial_+ \ln L \partial_+ X^\mu \\ D_{--} X^\mu &= \partial_{--} X^\mu - \partial_- \ln L \partial_- X^\mu \\ D_{+-} X^\mu &= D_{-+} X^\mu = \partial_{+-} X^\mu \\ D_{++-} X^\mu &= D_{+-+} X^\mu = \partial_{++-} X^\mu - \partial_+ \ln L \partial_{+-} X^\mu \\ D_{-++} X^\mu &= \partial_{-++} X^\mu - \partial_- (\partial_+ \ln L \partial_+ X^\mu) \end{aligned} \quad (31)$$

The last two of these equations show that i) just the number of \pm indices does not fully characterise a tensor; their order is important. ii) not all tensors with mixed indices are proportional to leading order EOM. The latter will alter the rules for constructing general actions in comparison to what was discussed in [3, 5]. However the last but one equation displays mixed-indices tensors that are indeed proportional to leading order EOM. This is a consequence of the following two important relations:

If $T_{\mu_1 \dots \mu_n}$ is a tensor with m_\pm indices of type \pm ,

$$\begin{aligned} D_+ T_{\mu_1 \dots \mu_n} &= \partial_+ T_{\mu_1 \dots \mu_n} - m_+ \partial_+ \ln L T_{\mu_1 \dots \mu_n} \\ D_- T_{\mu_1 \dots \mu_n} &= \partial_- T_{\mu_1 \dots \mu_n} - m_- \partial_- \ln L T_{\mu_1 \dots \mu_n} \end{aligned} \quad (32)$$

Hence covariant derivatives of tensors which are a combination of leading order EOM and its derivatives are also combinations of leading order EOM and its derivatives.

Another important property is that $D_{\pm\pm} X \cdot D_\pm X$ are linear combinations of leading order constraints $\partial_\pm X \cdot \partial_\pm X$ and their derivatives. That is,

$$D_{\pm\pm} X \cdot D_\pm X = \frac{1}{2} \partial_\pm (\partial_\pm X \cdot \partial_\pm X) - \partial_\pm \ln L (\partial_\pm X \cdot \partial_\pm X). \quad (33)$$

This too follows trivially from the second and third eqns of eqn.(31).

V. COVARIANT CALCULUS II: INTRINSIC METRIC AND WEYL SYMMETRY

In this section, we develop the covariant calculus based on the Polyakov approach which is both general coordinate invariant and Weyl-invariant.

We construct covariant derivatives with respect not only to the diffeomorphisms, but also to the Weyl-scaling symmetry, and use these objects to construct covariant terms. Although this approach is quite different from the non-intrinsic metric approach of section [IV], we will show in the end that the two approaches give identical results.

A. conformal symmetry

Beginning with the Polyakov action

$$S = \int d^2\sigma \sqrt{-h} h^{\alpha\beta} \partial_\alpha X \cdot \partial_\beta X \quad (34)$$

since X^μ is a worldsheet scalar, this construction ensures two-dimensional worldsheet reparametrisation invariance. The infinitesimal such transformation generated by $\sigma \rightarrow \sigma' = \sigma - \epsilon(\sigma)$ is given by

$$\delta_\epsilon X^\alpha = \epsilon^\gamma \partial_\gamma X^\alpha. \quad (35)$$

$$\delta_\epsilon h^{\alpha\beta} = \epsilon^\gamma \partial_\gamma h^{\alpha\beta} - \partial_\gamma \epsilon^\alpha h^{\gamma\beta} - \partial_\gamma \epsilon^\beta h^{\alpha\gamma}. \quad (36)$$

The important symmetry of (34) is of course the local Weyl Scaling, which only affects the metric,

$$h_{\alpha\beta} \rightarrow h'_{\alpha\beta} = \omega(\sigma) h_{\alpha\beta} \quad (37)$$

whose infinitesimal version with $\omega(\sigma) = 1 + \lambda(\sigma)$ reads

$$\delta_\lambda h_{\alpha\beta} = \lambda h_{\alpha\beta}. \quad (38)$$

A combination of the reparametrisation and Weyl symmetries is used to bring the worldsheet metric to the form $h_{\alpha\beta} = \eta_{\alpha\beta}$, called conformal gauge.

This choice of $h_{\alpha\beta}$ does not fix the coordinates and the freedom of Weyl scaling completely; A combined Weyl scaling and coordinate transformation such that

$$\lambda h_{\alpha\beta} = \partial_\beta \epsilon_\alpha + \partial_\alpha \epsilon_\beta \quad (39)$$

preserves $h_{\alpha\beta} = \eta_{\alpha\beta}$. This residual symmetry is worldsheet conformal symmetry. Defining coordinates $\tau^\pm = \tau \pm \sigma$, the remaining infinitesimal symmetries are parametrised by arbitrary functions

$$\epsilon^+(\tau^+), \quad \text{and} \quad \epsilon^-(\tau^-). \quad (40)$$

This is of course just the symmetry of eqn.(5).

B. Generalised Covariant Derivatives

What we need are quantities that transform covariantly under both general coordinate transformations as well as local Weyl scalings. Hence we need tensors with definite Weyl-scaling dimensions. A tensor ϕ of Weyl-scaling dimension j transforms under Weyl-scalings as

$$\phi \rightarrow \phi' = \omega(\sigma)^j \phi \quad (41)$$

The Weyl-weight of $h_{\alpha\beta}$ is 1 according to eqn.(37) (this is a matter of convention without any loss of generality). To see the issues involved, consider a worldsheet vector V_β with Weyl-weight j_V ; its covariant derivative with respect to reparametrisations is

$$\nabla_\alpha V_\beta = \partial_\alpha V_\beta - \Gamma_{\alpha\beta}^\gamma V_\gamma \quad (42)$$

with the connection $\Gamma_{\alpha\beta}^\gamma$ given by the standard Christoffel symbol

$$\Gamma_{\alpha\beta}^\gamma = -\frac{1}{2} h^{\gamma\delta} (\partial_\delta h_{\alpha\beta} + \partial_\alpha h_{\beta\delta} + \partial_\beta h_{\alpha\delta}). \quad (43)$$

Clearly under Weyl-scalings of V_α the covariant derivative of eqn.(42) does not scale in any simple way. In this particular example, there are two sources for this; the occurrence of the derivative of V on the one hand, and the occurrence of the derivatives of $h_{\alpha\beta}$ on the other. Ordinary derivatives of a tensor ϕ with definite Weyl-weight j do not simply scale when ϕ is locally scaled.

This motivates the definition of a new *Weyl-covariant derivative*; for a tensor field ϕ of Weyl-scaling dimension j , we set

$$\Delta_\alpha \phi \equiv \partial_\alpha \phi - j \chi_\alpha \phi \quad (44)$$

Restricting to the case when ϕ is a scalar, one sees that χ_α must transform as a worldsheet vector under reparametrisations. The Weyl-covariant derivative of a field with Weyl-scaling dimension j should again be a field with the same Weyl-scaling dimension j :

$$(\Delta_\alpha \phi)' = \omega^j \Delta_\alpha \phi \quad (45)$$

under the transformation (37). This requires the following inhomogeneous transformation of χ_α under Weyl-scaling

$$\chi'_\alpha = \chi_\alpha + \partial_\alpha \ln \omega \quad (46)$$

This immediately leads to the following generalisation of the Christoffel symbol that is appropriate for the present context:

$$\begin{aligned} G_{\alpha\beta}^\gamma &= \frac{1}{2} h^{\gamma\delta} (\Delta_\alpha h_{\beta\delta} + \Delta_\beta h_{\alpha\delta} - \Delta_\delta h_{\alpha\beta}) \\ &\equiv \Gamma_{\alpha\beta}^\gamma + W_{\alpha\beta}^\gamma \end{aligned} \quad (47)$$

where

$$W_{\alpha\beta}^\gamma = \frac{1}{2} (h_{\alpha\beta} \chi^\gamma - \delta_\alpha^\gamma \chi_\beta - \delta_\beta^\gamma \chi_\alpha). \quad (48)$$

From eqn.(47) it is easy to see that $G_{\alpha\beta}^\gamma$ is invariant under Weyl-scalings (it has Weyl-weight 0) while neither Γ nor W has well-defined Weyl-weight. Since $W_{\alpha\beta}^\gamma$ transforms as a proper tensor under reparametrisations, it follows that $G_{\alpha\beta}^\gamma$ also transforms as a proper connection. Putting these observations together, we define the Weyl-reparametrisation covariant derivative \mathcal{D}_α of a rank- n worldsheet tensor $T_{\beta_1 \dots \beta_n}$ of Weyl-scaling dimension j by

$$\begin{aligned} \mathcal{D}_\alpha T_{\beta_1 \dots \beta_n} &\equiv \Delta_\alpha T_{\beta_1 \dots \beta_n} \\ &\quad - G_{\alpha\beta_1}^\gamma T_{\gamma\beta_2 \dots \beta_n} \\ &\quad - \dots \\ &\quad - G_{\alpha\beta_n}^\gamma T_{\beta_1 \dots \beta_{n-1}\gamma} \end{aligned} \quad (49)$$

It is useful to rewrite this in the suggestive form

$$\begin{aligned}\mathcal{D}_\alpha T_{\beta_1 \dots \beta_n} &\equiv D_\alpha T_{\beta_1 \dots \beta_n} - j \chi_\alpha T_{\beta_1 \dots \beta_n} \\ &\quad - W_{\alpha\beta_1}^\gamma T_{\gamma\beta_2 \dots \beta_n} \\ &\quad - \dots \\ &\quad - W_{\alpha\beta_n}^\gamma T_{\beta_1 \dots \beta_{n-1}\gamma}\end{aligned}\quad (50)$$

In eqn.(49) every term has the same Weyl-weight as the tensor T and consequently so does $\mathcal{D}T$, but none of these terms transforms as a tensor under reparametrisations. On the other hand in eqn.(50) every term transforms as a tensor under reparametrisations while none of them has a definite Weyl-weight. Together equations (49) and (50) imply that $\mathcal{D}T$ is covariant under both Weyl-scalings and reparametrisations.

The various covariant derivatives obey a Leibniz rule, just as ∂_α does:

$$\nabla_\alpha(T_1 T_2) = \nabla_\alpha T_1 T_2 + T_1 \nabla_\alpha T_2, \quad (51)$$

$$\Delta_\alpha(T_1 T_2) = \Delta_\alpha T_1 T_2 + T_1 \Delta_\alpha T_2, \quad (52)$$

$$\mathcal{D}_\alpha(T_1 T_2) = \mathcal{D}_\alpha T_1 T_2 + T_1 \mathcal{D}_\alpha T_2. \quad (53)$$

where T_1 and T_2 are tensors, each with definite Weyl dimension, but not necessarily of the same rank.

\mathcal{D} sports the important property

$$\mathcal{D}_\alpha h_{\gamma\delta} = 0. \quad (54)$$

C. Weyl Connection

All the features discussed above hold for any choice of χ_α as long it responds to Weyl-scalings according to eqn.(46). In fact, according to that equation, a connection of the form

$$\chi_\alpha = \frac{1}{W_\Phi} \partial_\alpha \log \Phi \quad (55)$$

where Φ is any worldsheet *scalar* of Weyl-scaling dimension W_Φ , would be acceptable. It follows that

$$\mathcal{D}_\alpha \Phi = 0. \quad (56)$$

We shall choose Φ to be constructed from h and derivatives of X .

We are still free to choose a form for Φ . We are not constrained to use only one form for Φ ; anything will do so long as it is a scalar with non-zero Weyl-dimension, and also that it is conformity with the Denominator Principle of [III E]. This constrains Φ to be of the form

$$\Phi = \mathcal{L} + \text{higher order in } 1/R. \quad (57)$$

where

$$\mathcal{L} \equiv h^{\alpha\beta} \mathcal{D}_\alpha X \mathcal{D}_\beta X \equiv X_{;\alpha} X_{;\alpha} \quad (58)$$

By arguments identical to the ones that led to eqn.(17) as the simplest choice for the metric substitute in the first approach, we conclude that the simplest choice for Φ is

$$\Phi = \mathcal{L} \quad W_\Phi = -1 \quad (59)$$

In section [VI] we shall see that there is indeed an intimate connection between these two choices.

D. Manifestly Covariant Action Terms - II

After constructing all the Weyl-reparametrisation covariant derivatives $\mathcal{D}_{\alpha\beta\dots} X^\mu$ with Weyl-weight 0, the Weyl-reparametrisation covariant generalised Riemann tensor

$$\mathcal{R}_{\beta\gamma\delta}^\alpha \equiv \Delta_\gamma G_{\beta\delta}^\alpha - \Delta_\delta G_{\beta\gamma}^\alpha + G_{\gamma\eta}^\alpha G_{\delta\beta}^\eta - G_{\delta\eta}^\alpha G_{\gamma\beta}^\eta \quad (60)$$

and its Weyl-reparametrisation covariant derivatives, all of Weyl-weight 0, one can construct action integrands which are scalar densities under reparametrisation and invariant under Weyl-scalings. We shall do this as a two-step process to highlight important differences from the corresponding construction in section [IV A]; first we shall construct scalar densities under reparametrisation and use Weyl-scaling covariance to eventually obtain our quantities of interest. The first step is very similar to what was done in section [IV A]. Let us illustrate this by working out the analog of eqn.(25) of section [IV];

$$\bar{\mathcal{N}}_1 = \sqrt{h} \{ \mathcal{D}_{\alpha_1\beta_1} X \cdot \mathcal{D}_{\alpha_2\beta_2} X h^{\alpha_1\alpha_2} h^{\beta_1\beta_2} \}^2 \quad (61)$$

The Weyl-weight of $\bar{\mathcal{N}}_1$ is -3 and that brings us to the second step; in order to get a term of Weyl-weight 0 one has to multiply by something with Weyl-weight 3. Clearly there are many ways of doing so. We call these *Weyl-weight Compensators*. We now show that if the ‘total divergence’ property of covariant derivatives is to be extended to the Weyl-reparametrisation covariant derivatives, these compensators have to be appropriate powers of Φ of eqn.(55).

Consider a contravariant vector V^α of Weyl-weight J . Its Weyl-reparametrisation covariant derivative is given by

$$\mathcal{D}_\alpha V^\beta = \nabla_\alpha V^\beta - J \chi_\alpha V^\beta + W_{\alpha\gamma}^\beta V^\gamma. \quad (62)$$

Hence

$$\mathcal{D}_\alpha V^\alpha = \nabla_\alpha V^\alpha - J \chi_\alpha V^\alpha + W_{\alpha\gamma}^\alpha V^\gamma. \quad (63)$$

On recalling $\nabla_\alpha V^\alpha = \frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} V^\alpha)$ and $W_{\alpha\gamma}^\gamma = -\chi_\gamma$, one gets

$$\mathcal{D}_\alpha V^\alpha = \frac{\Phi^{\frac{J+1}{W_\Phi}}}{\sqrt{h}} \partial_\alpha (\sqrt{h} \Phi^{-\frac{J+1}{W_\Phi}} V^\alpha) \quad (64)$$

Thus in order to convert the scalar density $\sqrt{h} \mathcal{D}_\alpha V^\alpha$ of Weyl-weight $J+1$ into a scalar density with Weyl-weight 0 so that the total divergence property is maintained, it has to be multiplied only by $\Phi^{-(J+1)/W_\Phi}$ and not by just any expression with Weyl-weight $-(J+1)$. In other words, the Weyl-weight Compensators have to be appropriate powers of Φ . With the specific choice of eqn.(59)

these compensators are powers of \mathcal{L} . Thus the final desired expression for our example is

$$\mathcal{N}_1 = \sqrt{h} \mathcal{L}^{-3} \{ \mathcal{D}_{\alpha_1 \beta_1} X \cdot \mathcal{D}_{\alpha_2 \beta_2} X h^{\alpha_1 \alpha_2} h^{\beta_1 \beta_2} \}^2 \quad (65)$$

We will show later that eqn.(65) and eqn.(25) are the same.

E. Conformal Gauge and Conformal Transformations in Calculus-II

As explained in detail above, we begin with both Weyl and coordinate invariance and intend to fix both to end up with something written in “+/-” notation. The resultant actions will be invariant under the conformal transformation (5).

A choice of coordinates σ^\pm and Weyl scaling (37) is made to set

$$h_{+-} = h_{-+} = 2, \quad h^{+-} = h^{-+} = 1/2. \quad (66)$$

We write gauge-fixed quantities using a ‘check’ and covariant quantities in script letters. For example,

$$\mathcal{L} \equiv h^{\alpha\beta} \mathcal{D}_\alpha X \mathcal{D}_\beta X \equiv X_{;\alpha} X_{;\alpha} \rightarrow \check{\mathcal{L}} \sim L \equiv \partial_+ X \partial_- X. \quad (67)$$

here we write the covariant \mathcal{D} derivative with a “;” to save space, and also assume that repeated indices are summed using the metric.

In this gauge $\Gamma^{(2)}_{\alpha\beta} = 0$ and the W-tensor is given by

$$\check{W}_{\alpha\beta}^\gamma = -\frac{1}{2L} (h_{\alpha\beta} \partial^\gamma - \delta_\alpha^\gamma \partial_\beta - \delta_\beta^\gamma \partial_\alpha) L. \quad (68)$$

F. + and - skeletal forms

In this section we explore some of the consequences of this gauge fixing. Suppose $T_{\alpha\beta\dots}$ is a gauge-fixed tensor of Weyl-dimension j .

$$\mathcal{D}_+ T_{\dots}^{(j)} = \partial_+ T_{\dots} - j \chi_+ T_{\dots} - t_+ W_{++}^+ T_{\dots} \quad (69)$$

where we have used that $W_{++}^- = 0$ and $W_{+-}^\pm = 0$, and t_+ is the number of + indices on T . Evaluating the gauge-fixed W -connection, the only components which do not vanish are

$$W_{++}^+ = -\chi_+ = \partial_+ \ln L \quad W_{--}^- = -\chi_- = \partial_- \ln L \quad (70)$$

and thus

$$\mathcal{D}_+ T_{\dots}^{(j)} = \partial_+ T_{\dots} - j \chi_+ T_{\dots} + t_+ \chi_+ T_{\dots} \quad (71)$$

Similarly for $+ \leftrightarrow -$,

$$\mathcal{D}_- T_{\dots}^{(j)} = \partial_- T_{\dots} - j \chi_- T_{\dots} + t_- \chi_- T_{\dots} \quad (72)$$

Evidently,

$$[\mathcal{D}_+, \mathcal{D}_-] T_{\dots}^{(j)} = (t_- - t_+) (\partial_+ \chi_-) T_{\dots}^{(j)} \quad (73)$$

$$= (t_- - t_+) (\partial_+ \partial_- \log \Phi) T_{\dots}^{(j)}, \quad (74)$$

in fact consistent with our earlier calculations, despite the difference in formalism.

We now show that the Weyl-reparametrisation covariant derivatives $\mathcal{D}_{\alpha\beta\dots} X^\mu$ are identical to the covariant derivatives $D_{\alpha\beta\dots} X^\mu$ of section [IV]. We show this recursively by first proving that covariant derivatives of zero Weyl-weight tensors are the same in both methods. Consider such zero weight tensors $T_{\beta_1\dots\beta_n}$. Then

$$\begin{aligned} \mathcal{D}_\alpha T_{\beta_1\dots\beta_n} &= \partial_\alpha T_{\beta_1\dots\beta_n} - G_{\alpha\beta_1}^\gamma T_{\gamma\beta_2\dots} - \dots \\ &= \partial_\alpha T_{\beta_1\dots\beta_n} - \check{W}_{\alpha\beta_1}^\gamma T_{\gamma\beta_2\dots} - \dots \\ &= \partial_\alpha T_{\beta_1\dots\beta_n} - \Gamma_{\alpha\beta_1}^{(1)\gamma} T_{\gamma\beta_2\dots} - \dots \\ &= D_\alpha T_{\beta_1\dots\beta_n}. \end{aligned} \quad (75)$$

We have used the important fact that the components of gauge fixed W-tensor given by eqn.(70) are identical to those of the Christoffel connection of section [IV] given in eqn.(28) and that the Christoffel symbols $\Gamma^{(2)}$ of covariant calculus-II in its conformal gauge are all 0. In other words, the $G_{\alpha\beta}^\gamma$ is the same in the two conformal gauges. This is easily understood as the metric choices of eqn.(27) and eqn.(66) are related by the Weyl-scaling factor L , and the tensor $G_{\alpha\beta}^\gamma$ is itself of Weyl-weight 0.

An immediate and important corollary is that all the components of generalised Riemann tensor $\mathcal{R}_{\beta\gamma\delta}^\alpha$ of the second approach are identical to the standard Riemann tensor $R_{\beta\gamma\delta}^\alpha$ of section [IV]. In the light of eqn.(75) all the covariant derivatives of the Riemann tensors are also the same in the two approaches. This means that the tensor ingredients of the two approaches in their conformal gauges are the same. However, what are different are the metric tensors needed to construct scalar densities, and the compensators in the second approach. We shall however show in section [VI] that even these conspire to match perfectly. It is shown in that section that this is not just an accident of the choices made in eqn.(17) and eqn.(59) but is a more general feature. It is also shown in that section that the said equivalence continues to hold even when action terms are constructed using tensors of nonzero Weyl-weight.

VI. EQUIVALENCE OF THE TWO CONFORMAL GAUGE FORMALISMS

We shall now show that the two conformal gauge formalisms are equivalent and that this equivalence is more general than the explicit choices made in eqn.(17) and eqn.(59). We illustrate this equivalence by again considering the covariant actions of eqn.(25) and eqn.(65). We have already shown that all the covariant derivatives of X^μ are all the same. Evaluating eqn.(65) in the conformal gauge of eqn.(66) one gets

$$\check{\mathcal{N}}_1 = \frac{1}{2} L^{-3} (D_{++} X \cdot D_{--} X + D_{+-} X \cdot D_{+-} X)^2 \quad (76)$$

On the other hand evaluating eqn.(25) in the conformal gauge eqn.(27) one gets

$$\check{M}_2 = 4L^{-3}(D_{++}X \cdot D_{--}X + D_{+-}X \cdot D_{-+}X)^2 \quad (77)$$

Thus the two terms are equal modulo an irrelevant constant.

Instead of the special choices eqn.(17) and eqn.(59) consider the pair

$$g_{\alpha\beta} = g_{\alpha\beta}^* \quad \Phi^* = h^{\alpha\beta} g_{\alpha\beta}^* \quad W_{\Phi^*} = -1 \quad (78)$$

If g^* satisfies the Denominator Principle of [III E] so will Φ^* , and vice versa. Let us denote the corresponding Weyl connection by χ_α^* . Now consider the pair of conformal gauge metrics related by a Weyl-scaling

$$h'_{+-} = \Phi^* h_{+-} \quad (79)$$

If h_{+-} is the metric of eqn.(66), the metric h'_{+-} according to eqn.(79) is the metric substitute g_{+-}^* . The choices eqn.(17), eqn.(59), eqn.(27) and eqn.(66) are specific realisations of this general scheme. On using eqn.(46) one finds

$$\chi_{\alpha}^{*\prime} = 0 \quad (80)$$

Furthermore

$$G_{\alpha\beta}^{\prime\gamma} = G_{\alpha\beta}^{\gamma} \quad \check{W}_{\alpha\beta}^{\prime\gamma} = 0 \quad \Gamma_{\alpha\beta}^{(2)\prime\gamma} = \Gamma_{\alpha\beta}^{(1)\gamma} \quad (81)$$

and

$$\Gamma_{\alpha\beta}^{(2)\gamma} = 0 \quad \check{W}_{\alpha\beta}^{\gamma} = \Gamma_{\alpha\beta}^{(1)\gamma} \quad (82)$$

Now following the same strategy as in proving eqn.(75) one shows that the Weyl-reparametrisation covariant derivatives of zero weight tensors are identical to the covariant derivatives of [IV]. It is also easy to see that the way the metric factors and compensators matched in the example discussed earlier in this section continues to work even for the general case and also for any action term considered. This establishes the complete equivalence of the two formalisms as long as all tensors considered are of zero Weyl weight.

This equivalence continues to hold even when we construct actions with tensors of non-zero Weyl weights. Firstly, the Weyl weight compensators pick up an additional factor Φ^{*J} where J is the sum of the Weyl weights of all the tensor factors. In place of eqn.(75) one has, when the tensors are of non-zero weight,

$$\mathcal{D}_{\alpha_1 \dots \alpha_n} T_{\beta_1 \dots \beta_n} = \Phi^{*-j} D_{\alpha_1 \dots \alpha_n} T_{\beta_1 \dots \beta_n} \quad (83)$$

This results in an exact compensation of the j -dependent factors and one ends up with the equality of the action terms (modulo irrelevant constant factors) just as in the earlier case of the construction with zero weight tensors.

VII. SYSTEMATIC CONSTRUCTION OF EFFECTIVE ACTIONS IN CONFORMAL GAUGE

We have shown how to construct manifestly covariant action terms in both the approaches in sections [IV A] and [V D]. At a classical level this is all that is required. At a quantum level, one has to work with gauge-fixed actions. As long as the symmetries are not violated through quantum corrections, any gauge is as good as any other. We shall restrict ourselves to the conformal gauge and discuss the procedure for a systematic construction of effective actions. Nevertheless, often it is instructive to work in different gauges both because of technical simplicity as well as for demonstration of gauge invariance.

As we have already demonstrated the complete equivalence of the conformal gauges of the two approaches in [VI] we shall use the form of the results of section [IV B]; one could equally well have used section [V E].

The systematic construction of effective action terms that are manifestly covariant under conformal transformations proceeds more or less along the lines of what has already been presented in [3] with some improvements suggested in [5], but with some very important differences which we address here. Before that, we draw attention to the fact that these earlier methods were based on using skeletal forms which were ordinary derivatives of X^μ . Because of this the transformation laws that left these actions invariant had to be discovered each time, and by trial and error. Our constructions in this paper now allow the skeletal forms to be built out of covariant derivatives and because of this, invariance of the action terms is guaranteed.

As before the method of construction involves stringing together covariant derivatives of X^μ duly contracted with target space invariant tensors $\eta_{\mu\nu}, \epsilon_{\mu_1 \dots \mu_D}$ and then rendered into $(1,1)$ worldsheet tensors by dividing with appropriate powers of L . As before, terms proportional to the constraints and their derivatives are dropped. Covariantly this amounts to dropping terms proportional to $D_{\pm\pm}X \cdot D_{\pm\pm}X$ and their covariant derivatives.

The main difference from what was presented in [3, 5] comes in the treatment of terms proportional to EOM and its derivatives. There they were simply dropped. As shown in detail in [3] dropping such terms amounts to a field redefinition which can affect the transformation laws as well as the measure (in the path integral approach). The covariant calculi presented here are based on the fixed form of transformation laws eqn.(5). Therefore in the systematic construction of terms such EOM terms can not be dropped.

Hence mixed covariant derivative terms (in the sense of having both $+$ and $-$ indices) have to be considered in the general construction in contrast to [3, 5]. Even apart from the EOM issue, eqn.(31) shows that not all mixed covariant derivatives, unlike mixed ordinary derivatives, are proportional to EOM.

It was shown in [3] that as long as one is interested

in terms up to order R^{-3} , such field redefinitions can be safely carried out without worrying about the invariant measure or the Jacobians for transformation. The transformation laws, however, have to be modified. A practical way out of the latter is to first work out the full equations of motion and the full stress tensor for covariant actions constructed by our covariant calculus and then express these in terms of the new fields. Even when working with action term of higher than R^{-3} order, it may prove desirable from a calculational point of view to drop such EOM terms and carry out the concomitant changes. The details depend on the particular case at hand.

In the next two subsections we show how this systematic method may be applied at the level of the PS action terms as well as the Drummond terms at order R^{-6} .

As we shall see, the integrand of the PS term does not appear at all in the covariant formulation. In fact, it has to be treated and understood in a different way. The PS term is of course essential to ‘adjust’ the central charge of the theory; without the PS term the effective string construction is not consistent outside the usual critical dimension. We discuss this peculiar situation and the impossibility of covariantising the integrand of the PS term in section [VIII]. However, as noted there and as already known from earlier works, the action represented by the PS term is indeed conformally invariant.

A. Attempts at covariantising the PS Terms

In this subsection we make an attempt at covariantising the integrand of the PS term. As discussed at length in [3] there are two (in particular) equivalent forms for the PS term that differ by total derivative and EOM terms. These are

$$I_{PS}^{(1)} = \frac{1}{L} \partial_+^2 X \cdot \partial_-^2 X \quad (84)$$

and

$$I_{PS}^{(2)} = \frac{1}{L^2} \partial_+^2 X \cdot \partial_- X \partial_-^2 X \cdot \partial_+ X. \quad (85)$$

This second expression is the form given in [1], appearing in eqn.(1), and we generally refer to it as “the PS term”.

Let us consider the first of these. The obvious conformal-gauge candidate for this is

$$I_{PSConf}^{(1)} = L^{-1} D_{++} X \cdot D_{--} X. \quad (86)$$

On using eqn.(31) eqn.(86) can be expanded as

$$\begin{aligned} & \partial_+^2 X \cdot \partial_-^2 X - L^{-1} \partial_+ L \partial_- L \\ & + \frac{\partial_+ L \partial_- X \cdot \partial_+ X + \partial_- L \partial_+ X \cdot \partial_- X}{L}. \end{aligned} \quad (87)$$

On recalling the following identity from [3]

$$\begin{aligned} & \frac{\partial_+^2 X \cdot \partial_-^2 X}{L} = \frac{\partial_+^2 X \cdot \partial_- X \partial_-^2 X \cdot \partial_+ X}{L^2} \\ & + \frac{\partial_+ X \cdot \partial_+ X}{L} - \frac{\partial_+ X \cdot \partial_+ X \partial_+ X \cdot \partial_- X}{L^2} \\ & + \partial_- \left(\frac{\partial_+^2 X \cdot \partial_- X}{L} \right) - \partial_+ \left(\frac{\partial_+ X \cdot \partial_- X}{L} \right) \end{aligned} \quad (88)$$

we see that

$$L^{-1} D_{++} X \cdot D_{--} X = \text{Total Derivative} + \text{EOM} \quad (89)$$

Thus though eqn.(86) appears to be a candidate for covariant form of eqn.(84) it ends up being a linear combination of total derivative terms and EOM. Through a more tedious calculation it can be shown that the second term eqn.(85) meets the same fate. In fact, using the systematic procedure for constructing actions discussed above, it is easily seen that it is not possible to write any covariant term reproducing the PS terms. A clue to this ‘anomalous’ behaviour is already present in eqn.(7). We shall prove this impossibility in a different and more fundamental way in section [VIII].

B. Covariantising the Drummond Terms

Before proceeding, we make a few statements on terms proportional to EOM. At this order one has to explicitly verify whether EOM terms can be dropped or not and they can not be dropped generically. However, in [2] EOM terms were dropped in arriving at eqn.(20). Thus a comparison can only be made if we examine the terms modulo EOM, but otherwise we emphasise that the general construction proposed in this paper is the more legitimate.

Let us start with eqn.(24) and eqn.(25). It is easy to work out these expressions in the conformal gauge eqn.(27):

$$\begin{aligned} \mathcal{M}_1 &= 2 \frac{D_{++} X \cdot D_{++} X \cdot D_{--} X \cdot D_{--} X}{L^3} \\ &+ 2 \frac{(D_{++} X \cdot D_{--} X)^2}{L^3} \end{aligned} \quad (90)$$

$$\mathcal{M}_2 = \frac{4}{L^3} (D_{++} X \cdot D_{--} X)^2 \quad (91)$$

We consider the particular combination

$$\mathcal{M}_1 - \frac{\mathcal{M}_2}{2} = \frac{2}{L^3} (D_{++} X \cdot D_{++} X) (D_{--} X \cdot D_{--} X), \quad (92)$$

and it is easy to show that, modulo terms that are leading-order constraints and their derivatives, this is just \mathcal{M}_1 . To understand \mathcal{M}_2 let us display eqn.(87) slightly differently as

$$L^{-1} D_{++} X \cdot D_{--} X = \partial_- (L^{-1} \partial_+^2 X \cdot \partial_- X) + \text{EOM} \quad (93)$$

Then it follows that

$$\begin{aligned} M_2 &= L^{-1}[L^{-1}\partial_+^2 X \cdot \partial_-^2 X - L^{-2}\partial_- L \partial_+^2 X \cdot \partial_- X]^2 \\ &= M_2 - 2M_3 + M_4 \end{aligned} \quad (94)$$

This way we are able to obtain two independent linear combinations of eqn.(20). It can be shown, through straightforward but tedious algebra, that the covariant calculus can not produce any other combinations. The obvious approach to covariantising the eqn.(20) by replacing ordinary derivatives by covariant derivatives only produces, apart from these combinations, EOM and derivatives, constraints and their derivatives, and total derivatives. This is completely analogous to the situation with PS terms discussed in the previous section.

The present formalism, while representing quite a general way of formulating covariance, is thus extremely restrictive. The only possible gauge-fixed action to R^{-6} order is, up to irrelevant terms,

$$\int \frac{d^2\sigma}{4\pi} \left(\frac{L}{a^2} + \beta \frac{\partial_+ L \partial_- L}{L^2} + \beta_1(M_2 - 2M_3 + M_4) + \beta_2 M_1 \right). \quad (95)$$

Drummond's terms appear, but only in these particular combinations. This is one of the main results of the present paper; which we now emphasise. The effective string action has only three parameters β, β_1, β_2 (in addition to the string tension a^2 which merely sets a physical scale) at order R^{-6} . Of these the second term, which corresponds to the PS term, does not transform covariantly. Nevertheless, the PS action is invariant under the conformal transformations. Of the three parameters, the leading order analysis as given by PS already fixes β leaving only two free parameters β_1, β_2 . It would be interesting to see if higher order analysis would further fix some of the remaining parameters.

VIII. IMPOSSIBILITY OF COVARIANTISING THE PS INTEGRAND

We shall prove that for WZNW effective actions for a conformal anomaly in two dimensions defined by

$$\delta_\lambda S_{WZNW} = \int d^2\xi \lambda(\xi) \sqrt{g} R(\xi) \quad (96)$$

the integrand of S_{WZNW} can never be manifestly covariant under coordinate transformations.

Here, $R(\xi)$ is the Ricci scalar. The action proposed by Polyakov [6] in the context of two dimensional quantum gravity,

$$S_{\text{Polya}} = \int R \frac{1}{\nabla^2} R \quad (97)$$

is such a WZNW action.

Written out explicitly,

$$S_{\text{Polya}} = \int d^2x \sqrt{g(x)} R(x) \left(\frac{1}{\nabla^2} R \right)(x) \quad (98)$$

where $R(x)$ is the scalar curvature in two dimensions. The value of the integrand in the conformal gauge of eqn.(27) is

$$\begin{aligned} I_{\text{Polya}} &= (\partial_+ \partial_- \ln L) \cdot \frac{1}{\partial_+ \partial_-} \cdot (\partial_+ \partial_- \ln L) \\ &= (\partial_+ \partial_- \ln L) \ln L \end{aligned} \quad (99)$$

This is the same as the integrand L_2 of eqn.(6) upto total derivative terms. The variation of this under conformal transformations eqn.(5) is

$$\delta I_{\text{Polya}} = \partial_- (\epsilon^- I_{\text{Polya}}) + \partial_- \epsilon^- \partial_+ \partial_- \ln L \quad (100)$$

Since $\partial_+ \epsilon^- = 0$, it follows that the Polyakov action is indeed invariant under conformal transformations.

The Weyl scalings are defined by

$$\delta_{\lambda(\xi)} g_{\alpha\beta} = \lambda(\xi) g_{\alpha\beta} \quad (101)$$

and the infinitesimal coordinate transformations are given by

$$\delta_\epsilon g_{\alpha\beta} = \epsilon^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \epsilon^\gamma g_{\gamma\beta} + \partial_\beta \epsilon^\gamma g_{\alpha\gamma} \quad (102)$$

Now we look for a combination of Weyl scaling and coordinate transformation that leaves the form of the metric unchanged;

$$\bar{\lambda} g_{\alpha\beta} = \epsilon^\gamma \partial_\gamma g_{\alpha\beta} + \partial_\alpha \epsilon^\gamma g_{\gamma\beta} + \partial_\beta \epsilon^\gamma g_{\alpha\gamma}. \quad (103)$$

The strategy is to consider

$$\delta_{tot} = \delta_{\bar{\lambda}} - \delta_\epsilon, \quad (104)$$

and by construction

$$\delta_{tot} \tilde{\mathcal{L}} = 0. \quad (105)$$

Although we are talking about the same transformations we discussed in section [V A], it is worth emphasising that here eqn.(105) does not say anything about the invariance or lack of invariance of the action under any of the said transformations. On the other hand, we have, from eqn.(96)

$$\delta_{\text{Weyl}}(\bar{\lambda}) \tilde{\mathcal{L}} = \bar{\lambda} \sqrt{g} R(\xi) + \partial_+ X^+(g, \bar{\lambda}) + \partial_- X^-(g, \bar{\lambda}) \quad (106)$$

It should be noted that the dependence of X^\pm is explicitly on $\bar{\lambda}$ and its derivatives.

For the remainder we work explicitly in the conformal gauge of eqn.(27) and without loss of generality, restrict our attention to only ξ^- diffeomorphisms ($\epsilon^+ = 0$). In this case eqn.(103) reads

$$\partial_+ \epsilon^- = 0; \quad \bar{\lambda} = \partial_- \epsilon^- + \epsilon^- \partial_- \ln L \quad (107)$$

Using eqn.(30) and eqn.(107) we rewrite eqn.(106) as

$$\begin{aligned} \delta_{\text{Weyl}}(\bar{\lambda}) \tilde{\mathcal{L}} &= \partial_- \{-2\epsilon^- \partial_+ \ln L + X^-\} \\ &+ \partial_+ \{2\epsilon^- \partial_- \ln L - \epsilon^- (\partial_- \ln L)^2 + X^+\} \end{aligned} \quad (108)$$

Now, this must equal $\delta(\epsilon^-)\tilde{\mathcal{L}}$. If $\tilde{\mathcal{L}}$ were transforming as a scalar density, the ∂_+ terms in the last line of eqn.(108) must vanish identically. This can happen only if

$$\partial_+ X^+ = \partial_+ \{ \epsilon^- (-2\partial_- \ln L + (\partial_- \ln L)^2) \} \quad (109)$$

As we have already noted X^+ must have an explicit dependence on $\tilde{\lambda}$ and its derivatives. Part of eqn.(109) can indeed be cast into this form (which part can be so cast is not unique);

$$\begin{aligned} \partial_+ X^+ &= \partial_+ \{ -2\partial_- (\epsilon^- \partial_- \ln L + \partial_- \epsilon^-) \\ &\quad + (\epsilon^- \partial_- \ln L + \partial_- \epsilon^-) \partial_- \ln L + \partial_- \epsilon^- \partial_- \ln L \}. \end{aligned} \quad (110)$$

The last term $\partial_- \epsilon^- \cdot \partial_+ \ln L$ makes it evident that $\tilde{\mathcal{L}}$ fails to be a scalar density. It fails by precisely the same term as obtained through explicit calculations with I_{Polya} , which we have seen in eqn.(100). Nevertheless, S_{Polya} is invariant as shown above.

IX. CONCLUSIONS

We have given in this paper a vastly simplified approach to the theory of effective strings in comparison with the PS formalism. The essential simplification is that the transformation law is always the same as the standard transformation law for free bosonic string action. The transformation law does not have to be tuned to the action. In the conformal gauge, these represent conformal transformations exactly, and not approximately as in the case of the PS transformation law (only to order R^{-3} to be precise).

Consequently every action constructed by our covariant calculus, and in particular S_2 of eqn.(4), is in principle valid to *all orders* in R^{-1} . Whether phenomenologically any action or combination of actions is correct or not is a different issue.

A further consequence of this covariantisation is the restriction of the effective action to order R^{-6} to include only two free parameters β_1 and β_2 . Our result for the complete effective action to this order, from our first approach, is the truncation to order R^{-6} of

$$\begin{aligned} S &= \frac{\beta}{4\pi} S_{\text{Polya}} + \int \frac{d^2\sigma}{4\pi} \left[\frac{\sqrt{g}}{a^2} + \beta_1 X^{;\alpha\beta} \cdot X^{;\delta\gamma} X_{;\alpha\beta} \cdot X_{;\delta\gamma} \right. \\ &\quad \left. + \beta_2 X^{;\alpha\beta} \cdot X_{;\alpha\beta} X^{;\gamma\delta} \cdot X_{;\gamma\delta} \right], \end{aligned} \quad (111)$$

The conformal gauge expression for eqn.(111) is given by eqn.(95) and that entire action is by construction *exactly* conformally invariant under the transformation law (5). Both the approaches yield the same actions in the conformal gauge. As already emphasised before, one can use the entire eqn.(95), without truncation, if one so wishes. Further terms presumably begin to appear at $\mathcal{O}(R^{-8})$.

Finally, it is worth emphasising that of course without a covariant formulation, one can only identify the relevant terms which may be included in the action, up to a given order, and adorn these terms with coefficients which are then the parameters of the theory. These must then be determined phenomenologically (using lattice QCD, for example). In contrast, in our covariant construction, in general the number of new free parameters introduced at each order in $1/R$ is fewer than the number which would obtain given such independent insertion of all relevant terms. This highly desirable reduction in parameters is also a conclusion which could be subject to verification in simulations or experiments.

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